Lorentz Manifolds: A Topological Study

A Synopsis

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1 Introduction

Topology concerns with the properties of space that are preserved under continuous deformations including stretching and bending, but not tearing or gluing. The term *topology* was introduced by *Johann Benedict Listing* in the 19th century, although it was not until the first decades of the 20th century that the idea of a topological space was developed. Henri Poincaré introduced the concepts of homotopy and homology, which are now considered parts of algebraic topology. The motivating insight behind topology is that some geometric problems depend not on the exact shape of the objects involved, but rather on the way they are put together. This generates the notion of homeomorphism. Homeomorphism can be considered as the most basic topological equivalence [11].

Special Relativity is the accepted physical theory regarding the relationship between space and time. It was originally proposed in 1905 by Albert Einstein. Einstein eventually identified the property of spacetime which is responsible for gravity as its curvature. In general relativity, it is assumed that space-time is curved by the presence of matter (energy), this curvature being represented by the Riemann tensor. Space and time in Einstein's universe are no longer flat but can be pushed and pulled, stretched and warped by matter. Gravity depends on the curvature of the space. Einstein derived special relativity formally from the assumption that the speed of light is the same to all observers, Hermann Minkowski built on mathematical approaches used in non-Euclidean geometry and the mathematical work of Lorentz and Poincaré. Minkowski showed in 1908 that Einstein's new theory could also be explained by replacing the concept of a separate space and time with a four-dimensional continuum called space-time. This was a groundbreaking concept, and Roger Penrose has said that relativity was not truly complete until Minkowski reformulated Einstein's work [19].

The concept of a four-dimensional space is hard to visualise. It may help at the beginning to think simply in terms of coordinates. In three-dimensional space, one needs three real numbers to refer to a point. German mathematician Minkowski realized that Einstein′s theory of Special Relativity can be appreciated in a space-time defined using Lorentz inner product. This four dimensional space-time was named as *Minkowski space*. In the Minkowski space, one needs four real numbers (three space coordinates and

one time coordinate) to refer to a point at a particular instant of time [32]. Let M denote Minkowski space, the real 4-dimensional space-time continuum of special relativity. It is customary to think of M as having the topology of real 4-dimensional Euclidean space, although there are reasons why this is wrong. In particular: (1) The 4-dimensional Euclidean topology is locally homogeneous, whereas M is not; every point has associated with it a light cone separating space vectors from time vectors. (2) The group of all homeomorphisms of 4-dimensional Euclidean space is vast, and of no physical significance. Zeeman defined physically relevant non-Euclidean topologies on the Minkowski space [37].

Manifolds play a decisive role in Einstein's General Theory of Relativity, which describes the interactions among matter, energy, and gravitational forces. The central assertion of the theory is that space-time can be modeled by a 4-dimensional manifold that carries a certain kind of geometric structure called a Lorentz metric; and this metric satisfies a system of partial differential equations called the Einstein Field Equations. Gravitational effects are then interpreted as manifestations of the curvature of the Lorentz metric. There are especially interesting relationships between the local geometry of space-time (as reflected in the local distribution of matter and energy) and the global topological structure of the universe; these relationships are similar to those described above for Riemannian manifolds, but are more complicated because of the introduction of forces and motion into the picture [10].

One of the most fundamental and perplexing challenges for modern physics is to resolve the incompatibilities between Quantum Theory and General Relativity. An approach that some physicists consider very promising is called String Theory, in which manifolds appear in several different important roles. Several different kinds of String Theory have been constructed, but all of them give consistent results only if the cross-section is a certain kind of 6-dimensional manifold known as a *CalabiYau manifold*. More recently, evidence has been uncovered that all of these string theories are different limiting cases of a single underlying theory, dubbed *M-Theory*, in which the cross-section is a 7-manifold. These developments in physics have stimulated profound advancements in the mathematical understanding of manifolds of dimensions 6, 7 and *CalabiYau manifolds* in particular. Another role that manifolds play in String Theory is in describing the history of an elementary

particle. As a string moves through space-time, it traces out a 2-dimensional manifold called its *World Sheet* [10].

Most of the Topological Field Theories are defined on the flat Minkowski space-time used in particle physics. Minkowski space can be contracted to a point, so a *Topological Quantum Field Theory* on Minkowski space computes only trivial topological invariants. Consequently, Topological Quantum Field Theories are usually studied on curved spacetimes such as Riemann surfaces. *Quantum Field Theory* in curved space-time is an extension of standard Quantum Field Theory on Minkowski space by writing it in covariant form [42].

The n-dimensional De Sitter space is the analog of n-sphere in Minkowski space. The De Sitter space was discovered by *Willem De Sitter* and at the same time, independently by Tullio Levi-Civita. In the context of General Relativity, classical study of cosmological constant tries to understand the geometry of De Sitter space-time [22]. The De Sitter space is of physical significance due to the fact that this is one of the solutions of Einstein Equations with a positive cosmological constant [21].

There are several physically significant non-manifold topologies on Minkowski space [37]. A study on these topologies has been undertaken by Zeeman [37], Nanda [27, 26], Doseena [12], Agrawal et al. [2, 3, 1, 4] etc. The present proposal is focused on a further exploration of these non-manifold topologies.

2 Notation and Preliminaries

In this section, we present the notations, basic definitions and known results which are used later. For terminologies which are not described here, we refer to [25, 8, 30, 7, 6, 13, 14, 20]. The symbols R and N denote the set of real and natural numbers respectively.

2.1 Minkowski Space

An *n-dimensional Minkowski space M* for $n \in N$ and $n > 1$, is the n-dimensional real vector space R^n with *Lorentz inner product*, where Lorentz

inner product is a bilinear form $g : R^n \times R^n \longrightarrow R$ satisfying the following properties (i) symmetric, i.e. $g(x, y) = g(y, x)$ for all $x, y \in R^n$ (ii) non-degenerate, i.e. if $g(x, y) = 0$ for all $y \in R^n$ then $x = 0$ and (iii) there exists a basis $\{e_0, e_1, ..., e_{n-1}\}$ of R^n such that $g(e_i, e_j) \equiv \eta_{ij}$ =1, for $i = j = 0$; -1, for $i = j = 1, ..., n-1$ and 0 otherwise.

2.1.1 Timelike, Lightlike and Spacelike vectors

An element $x \in M$ is called an *event*. For an event, $x \equiv \sum_{i=0}^{n-1} x_i e_i$, the coordinate x_0 is called *time component* and the coordinates $x_1, ..., x_{n-1}$ are called *spatial component* of x relative to the basis $\{e_0, e_1, ..., e_{n-1}\}$. Lorentz inner product induces an indefinite characteristic quadratic form Q on M defined by $Q(x) = g(x, x)$. Thus, $Q(x) = x_o^2 - \sum_{i=1}^{n-1} x_i^2$. An event $x \in M$ is called *timelike*, *lightlike* (*null*) or *spacelike* according as $Q(x)$ is positive, zero or negative. A straight line is called a *timelike, lightlike or spacelike* according as it is parallel to a timelike, lightlike or a spacelike vector.

2.1.2 Time cone, Light cone and Space cone

The sets $C^T(x) = \{\ y \in M : y = x \text{ or } Q(y - x) > 0\ \}$, $C^L(x) = \{\ y \in M : Q(y - x) = 0\ \}$ and $C^S(x) = \{\ y \in M : y = x \text{ or } Q(y - x) < 0\ \}$ are called the *time cone, light cone (or null cone), and space cone* respectively at the point $x \in M$. Further, the sets $C^+(x) \equiv \{y \in C^T(x) : y_0 > x_0\}$ and $C^-(x) \equiv \{y \in C^T(x) : y_0 < x_0\}$ are called the *future time cone* and *past time cone* at $x \in M$ respectively.

2.2 De sitter space

A D-dimensional De Sitter space is an embedding of the hypersurface with equation $dS_D = \{x \in M^{D+1}, x_o^2 - \sum_{i=1}^{D} x_i^2 = -R^2\}$ in $D + 1$ dimensional flat Minkowski space. It has constant negative curvature. The time x_0 runs vertically, while slices of x_0 are $D-1$ dimensional sphere S^{D-1} that contracts to reach a minimal radius and then re-expands. In other words, Topologically dS_D is $R \times S^{D-1}$ [21].

2.3 Topologies on Minkowski Space

2.3.1 Euclidean topology

The *Euclidean topology* on Minkowski space M is the topology generated by the basis $B = \{N_\epsilon^E(x) : \epsilon > 0, x \in M\}$ where N_ϵ^E denotes the Euclidean neighbourhood.

2.3.2 f-topology, Fine topology

The topology which is generated by the basis $\{N_\epsilon^E(x) \cap (C^T(x) \cup C^S(x)) : \epsilon > 0,\ x \in M\}$ is called the *f-topology* [1].
The *fine topology* on Minkowski space is the finest topology which induces Euclidean topology on every timelike straight line and spacelike hyperplane [37].

2.3.3 s-topology, Space topology

The topology which is generated by the basis $\{N_\epsilon^E(x) \cap C^S(x) : \epsilon > 0, x \in M\}$ is called the *s-topology* [27].
The *space topology* is the finest topology that induces Euclidean topology on every spacelike hyperplane [26].

2.3.4 t-topology, Time topology

The topology which is generated by the basis $\{N_\epsilon^E(x) \cap C^T(x) : \epsilon > 0, x \in M\}$ is called the *t-topology*[27].
The *time topology* is the finest topology that induces Euclidean topology on every timelike straight line [27].

2.3.5 A-topology

The *A -topology* is the finest topology on Minkowski space which induces Euclidean topology on every timelike and lightlike line and spacelike hyperplanes [29].

2.4 Manifolds

2.4.1 C^∞ Manifold

A topological space is called a *locally Euclidean space of dimension* n, if its every element has a neighborhood U homeomorphic to an open subset of R^n under a homeomorphism ϕ. The pair (U, ϕ) is called a coordinate neighborhood. A second countable Hausdorff space that is locally Euclidean of dimension n together with a C^∞ structure is called a C^∞ *manifold*, denoted by S [6].

2.4.2 Tangent Space

Let $C^\infty(p)$ denote the set of all C^∞ mappings at $p \in S$. Then the *tangent space* $T_p(S)$ at p is defined as the set of all mappings $X_p : C^\infty(p) \longrightarrow R$ satisfying for all $\alpha, \beta \in R$ and $f, g \in C^\infty(p)$ the two conditions (i) $X_p(\alpha(f) + \beta(g)) = \alpha(X_p f) + \beta(X_p g)$ and (ii) $X_p(fg) = (X_p f)g(p) + f(p)(X_p g)$ with the vector space operations in $T_p(S)$ defined by $(X_p + Y_p)f = X_p f + Y_p f$ and $(\alpha X_p)f = \alpha(X_p f)$. Every $X_p \in T_p(S)$ is called a tangent vector at $p \in S$ [6].

2.4.3 Geodesic

A curve $\alpha : (0, 1) \to S$ is said to be a *geodesic* if its covariant derivative is zero at each $t \in (0, 1)$. Let (U, ϕ) be a coordinate neighborhood of S and $q \in U$. Then there exist a neighborhood V of q and an $\epsilon > 0$ such that if $p \in V$ and $X_p \in T_p(S)$ with $\|X_p\| < \epsilon$, then there exist a unique geodesic $\gamma(t)$ defined for $-2 < t < 2$ and with $\gamma(0) = p, (d\gamma/dt)_{t=0} = X_p$.

2.4.4 Exponential Map

The map $exp_p : T_p(S) \longrightarrow S$ such that $exp_p(X_p) = \gamma(1)$ is called the exponential mapping, where γ is a unique geodesic determined by X_p. There exists a neighbourhood N of the origin of $T_p(S)$ such that the diffeomorphic image $U \equiv exp(N)$ is an open convex normal neighbourhood of p. The coordinate of each point $q \in U$ are the components of $X_q \in N$ with respect to some orthonormal basis of $T_p(S)$. These coordinates are called normal coordinates of $q \in U$.

2.4.5 Lorentz Manifold

A connected, Hausdorff, C^∞, real, n-dimensional manifold without boundary and with C^∞ Lorentz metric is called an *n-dimensional Lorentz manifold*. The 4-dimensional Lorentz manifold is called *space-time*. Throughout this section, an n-dimensional Lorentz manifold will be denoted by L [13].

2.4.6 Time Cone, Space Cone and Light Cone

Let $T_p(L)$ denote the tangent space of $p \in L$, U be a normal neighborhood of p, $exp : T_p(L) \longrightarrow L$ be the exponential mapping and $x^0(q), x^1(q) \dots x^{n-1}(q)$ denote the normal coordinates of $q \in U$. Then $x^i(p) = 0$, for $i = 0, 1, \dots, n-1$. The set $C^{T+}(p, U) = \{q \in U : (x^0(q))^2 - (x^1(q))^2 - \dots - (x^{n-1}(q))^2 > 0, x^0(q) > 0\}$ is called the *future time cone* at p and the set $C^{T-}(p, U) = \{q \in U : (x^0(q))^2 - (x^1(q))^2 - \dots - (x^{n-1}(q))^2 > 0, x^0(q) < 0\}$ is called the *past time cone* at p. The set $C^T(p, U) \equiv C^{T+}(p, U) \cup C^{T-}(p, U) \cup \{p\}$ is called *time cone* at p. Further, the sets $C^S(p, U) \equiv \{q \in U : (x^0(q))^2 - (x^1(q))^2 - \dots - (x^{n-1}(q))^2 < 0\}$ and $C^L(p, U) \equiv \{q \in U : (x^0(q))^2 - (x^1(q))^2 - \dots - (x^{n-1}(q))^2 = 0\}$ are called the *space cone* and *light cone* at p respectively [20].

2.4.7 Timelike curve and Spacelike curve

A curve $\gamma : [0, 1] \to L$ is called a *future-timelike curve* (or a *past-timelike curve* respectively) at $t_0 \in [0, 1]$ if γ is continuous and there is a connected neighborhood V of $t_0 \in [0, 1]$ and an open convex normal neighborhood U of $p \equiv \gamma(t_0)$ in L such that if $t_1 < t_0 < t_2$, then implies $\gamma(t_1) \in C^{T-}(p, U)$ and $\gamma(t_2) \in C^{T+}(p, U)$ (or $\gamma(t_2) \in C^{T-}(p, U)$ and $\gamma(t_1) \in C^{T+}(p, U)$ respectively), for every $t_1, t_2 \in V$ [20].
Let J be an interval in R. Then a curve $\gamma : J \to L$ is called a *spacelike curve* in a Lorentz manifold L and abbreviated as $\gamma \in \sum$, if γ is continuous and for every $t \in J$ there exists an open and connected neighbourhood W of $t \in J$ and a $U \in N(\gamma(t))$, such that $\gamma(W) \subseteq S(\gamma(t), U)$. If $\gamma \in \sum$ and J is compact, then γ is called a spacelike path in (S, g).

2.5 Topologies on Lorentz Manifolds

Path Topology

The path topology on Lorentz manifold L is defined as the finest topology which induces manifold topology on every timelike curve [20].

Zeeman Topology

It is defined as the finest topology that induces manifold topology on every timelike geodesic and spacelike hypersurface [16].

Space Topology

The space topology on Lorentz manifold L is defined as the finest topology which induces manifold topology on every spacelike curve [14].

2.6 Homotopy

2.6.1 Nulhomotopic

Let f and f' are continuous maps of the space X into the space Y. Then f is said to *homotopic to* f' (denoted as $f \simeq f'$) if there is a continuous map $F : X \times I \longrightarrow Y$ such that $F(x, 0) = f(x)$ and $F(x, 1) = f'(x)$ for each $x($ I=[0 1]). The map F is called *homotopy between* f *and* f'. Also, if $f \simeq f'$ and f' is a constant map, then f is *nulhomotopic* [25].

2.6.2 Retraction

If $A \subset X$, a *retraction* of X onto A is a continuous map $r : X \longrightarrow A$ such that $r|_A$ is the identity map of A. If such a map r exists, then A is called a *retract* of X [7].

2.6.3 Antipode Preserving map

If x is a point of S^n, then its antipode is the point $-x$. A map $h : S^n \longrightarrow S^m$ is said to be *antipode preserving* if $h(-x) = -h(x)$ $\forall x \in S^n$ [25].

2.7 Homology

2.7.1 p-Simplex

p-simplex s in R^n is the convex hull of collection of $p+1$ points $\{x_0, x_1, ...x_p\}$ in which $x_1 - x_0, ...x_p - x_0$ form linearly independent set [8].

2.7.2 Standard p-Simplex

Standard p-simplex σ_p is the set of all points $(t_0, t_1, .., t_p) \in R^{p+1}$ with $\sum t_i = 1$ and $t_i \geq 0$ for each i [8].

2.7.3 Singular p-Simplex

Let X be topological space. A *singular p-simplex* in X is a continuous function $\phi : \sigma_p \longrightarrow X$ [8].

2.7.4 Set of Singular n-Simplices

$S_n(X)$ is the free abelian group whose basis is the set of all singular n-simplices. An element of $S_n(X)$ has the form $\sum_\phi n_\phi \phi$ where n_ϕ is an integer, equal to zero for all but a finite number of ϕ [8].

2.7.5 Boundary Operator

A homomorphism $\delta : S_n(X) \longrightarrow S_{n-1}(X)$ is defined by $\delta = \delta_0 - \delta_1 + ... + (-1)^n \delta_n$, where $\delta_i : S_n(X) \longrightarrow S_{n-1}(X)$ is defined by $\delta_i(\sum_\phi n_\phi \phi) = \sum n_\phi \delta_i \phi$ [9, 8].

2.7.6 Singular Homology Group

An element $c \in S_n(X)$ is an *n-cycle* if $\delta(c) = 0$. An element $d \in S_n(X)$ is an *n-boundary* if $d = \delta(e)$ for some $e \in S_{n+1}(X)$. The Set of all n-cycles is a subgroup of $S_n(X)$, denoted by $Z_n(X)$. Similarily, the set of all n-boundaries is a subgroup of $S_n(X)$ and of $Z_n(X)$, denoted by $B_n(X)$. Thus *nth singular homology group* of a topological space X is the quotient group $Z_n(X)/B_n(X)$ which is denoted by $H_n(X)$ [9, 8].

2.8 Major Results

2.8.1 Sequence Lemma

Let X be a topological space and let $A \subseteq X$. If there is a sequence of points of A converging to x, then $x \in \overline{A}$. Conversely, if X is metrizable or first countable space then $x \in \overline{A}$ implies that there is a sequence of points of A converging to x [25].

2.8.2 Borsuk-Ulam Theorem

If $f : S^2 \longrightarrow R^2$ is a continuous map, then there is a point x of S^2 such that $f(x) = f(-x)$ [25].

2.8.3 Brouwer Fixed-Point Theorem

If $f : B^2 \longrightarrow B^2$ is a continuous map, then there exists a point x of B^2 such that $f(x) = x$ [25].

2.8.4 No-Retraction theorem

There is no retraction of B^2 onto S^1 [25].

2.8.5 Results on Homotopy and Homology Group

1. The inclusion map $j : S^1 \longrightarrow R^2 - 0$ is not nulhomotopic. The identity map $i : S^1 \longrightarrow S^1$ is not nulhomotopic [25].

2. If $h : S^1 \longrightarrow S^1$ is continuous and antipode-preserving, then h is not nulhomotopic [25].

3. There is no continuous antipode-preserving map $g : S^2 \longrightarrow S^1$ [25].

4. If X is a convex subset of R^n, then $H_p(X)$=0, for $p > 0$ [8].

5. If X is a non-empty pathwise connected space , then $H_0(X) \approx Z$ [8].

6. If U is an open set of R^n, then $H_p(U)$=0 for $p \geq n$ [9, 8].

7. Homology group of S^1 is $H_i(S^1) \approx Z$, for i=0,1 and $H_p(S^1) = 0$, for $p > 1$ [9, 8].

3 Survey of the Relevant Literature

In this section, a brief literature review has been carried out. In the beginning, the study on Zeeman's non-Euclidean topologies has been discussed. Thereafter, the work on generalized topologies on Lorentz manifolds has been presented.

In 1967, Zeeman [37] defined some physically important non-Euclidean topologies on Minkowski space namely fine topology, t-topology, s-topology, f-topology, space topology and time topology. The fine topology is one of the first non-Euclidean topology on Minkowski space. Zeeman obtained its homeomorphism group i.e. the group generated by Lorentz group, dilatations and translations and defined the notion of Zeno sequence on Minkowski space with fine topology. Dossena [12] studied compact sets of Minkowski space with the fine topology and proved that it is Hausdorff, path connected, separable, non-normal, non-locally compact, non-Lindelöf and non-first countable space and it is non-simply connected space for two-dimensional case. Popvassilev [31] determined that Minkowski space with fine topology is non-regular.

Zeeman conjectured that the homeomorphism group of the other non-Euclidean topologies is same as the homeomorphism group of fine topology. In 1971, Nanda proved that the homeomorphism group of space topology [26] is same as the homeomorphism group of fine topology and obtained that Minkowski space with space topology is non-normal, non-locally compact and non-second countable.

In 1972, Nanda showed that the Zeeman's conjecture is true for s-topology [27]. Further, s-topology is studied by Gunjan Agrawal and Sampada [3]. They studied the compact and connected sets of n-dimensional Minkowski space with s-topology. Also, it is found to be separable, first countable, path-connected non-regular, non-metrizable, non-second countable, non-compact and non-Lindelöf space. It is also obtained that 2-dimensional Minkowski space with s-topology is not simply connected [3]. Ersoy investigated the interior and closure of the space cone with respect to Euclidean topology and s-topology [33].

Nanda showed that the homeomorphism group of Minkowski space with t-topology is the group generated by Lorentz group, dilatations and transla-

tions [27]. Further, Minkowski space with t-topology is studied by Agrawal et al. [2]. Agrawal et al.[2] obtained that it is Hausdorff, path connected, separable, first countable, non-second countable, non-countably compact, non-Lindelf, non-regular, non-normal, non-locally compact, non-paracompact and non-metrizable. Also, they obtained the compact sets of Minkowski space with t-topology and proved that 2-dimensional Minkowski space with t-topology is non-simply connected.

Recently, Agrawal et al. proved that the invariance of domain theorem is not true in the context of t-topology. Also, a consequence that the no non-empty open subset of the n-dimensional Minkowski space with t-topology is homeomorphic to an open subset of the m-dimensional Minkowski space with t-topology has been proved[5]. In 2015, the homeomorphism group of Minkowski space with time topology is obtained by Agrawal et al. [4].

Agrawal et al.[1] obtained the characterization of compact sets of Minkowski space with the f-topology using the notion of Zeno sequences. It is proved that Minkowski space with the f-topology is Hausdorff, path connected, separable, first countable, non-regular, non-second countable, non-paracompact, non-Lindelöf, non-normal, non-locally compact and non-metrizable space. Also, two dimensional Minkowski space with f-topology is non-simply connected space.

Nanda [29] defined A-topology on Minkowski space and obtained the homeomorphism group of Minkowski space with A-topology i.e. the group generated by the Lorentz group, dialatations and translations. Nanda and Panda [28] studied a topology on Minkowski space as the topology generated by the positive cone at origin and its translates and called it as the K-topology or order topology. The order topology is not T_1, not Hausdorff, not compact, although it is locally compact, locally connected and pathwise connected. Further, it is simply connected and every loop based at a point is homotopic to the constant loop at that point. The homeomorphism group of order topology is the set of elements which are order automorphisms.

It has been found that there are generalizations of Zeeman's non-Euclidean topologies on Minkowski space to Lorentz manifold. Manifold topology is the most natural topology on Lorentz manifolds, although it is not physically significant. There are some physically significant non-manifold topologies which

include Zeeman topology, path topology, space topology, geodesic topology etc. Göbel [16] defined the Zeeman topology as a generalization of Zeeman's fine topology to Lorentz manifolds and studied the homeomorphism group of Zeeman topology i.e. the group of all homothetic transformations on space-time.

S.W. Hawking et al. [20] defined more appealing, physical and manageable path topology for strongly causal space-times. They obtained that it is first countable, separable, Hausdorff, path connected, locally path connected, although it is not regular, not normal, not locally compact and not paracompact. Further, it determines the causal, differential and conformal structure of space-time. Furthermore, this result is proved for every space-time by Malamant [24]. Also, Sumati Surya [41] obtained that causal bijections cannot exist between the space-times with different dimensions ($d > 2$) and causal structure determines the space-time dimension. It has been proved that path topology on Minkowski space is equivalent to t-topology on Minkowski space [30]. Low [23] showed that the generalization of t-topology to Lorentz manifolds is simply connected.

Göbel [17] defined different non-manifold topologies on Lorentz manifolds and conjectured that these are Hausdorff, locally path connected, path connected non-regular, non-normal and non-locally compact. The generalization of time topology to Lorentz manifolds is one of these topology.

Göbel [18] generalized the s-topology to Lorentz manifolds and named it as space topology and determined that the homeomorphism group of space topology with respect to time orientable strongly causal Lorentz manifolds is the group of all conformal C^∞ diffeomorphisms. Furthermore, it is conjectured that this result holds if there is no condition of strong causality of space-time. In 1983, Domiaty [13] proved this conjecture and again conjectured that the homeomorphism group of this topology remains same if space-time is not time orientable. In 1984, Domiaty [14] determined that the generalization of s-topology to Lorentz manifolds is 0-semimetrizable and conjectured space topology to be Hausdorff, first countable, locally path connected, path connected, non-regular.

Fullwood [15] proposed a new topology on space-time by combining idea of Hawking and Alexandrov. It has been proved that the distinguishing con-

dition holds on a manifold if and only if this new topology is equivalent to the path topology of Hawking, King and McCarthy. It is found that the homeomorphism group of this topology is the group of chronological and antichronological bijections in the case that the chronology condition holds on the space-time manifold. Hence, he found that this topology determines the causal, differential and conformal structure of space-time if space-time is distinguishable.

Hawking [38] investigated certain aspects of the geometry of the space-time manifold in the General Theory of Relativity with particular reference to the occurrence of singularities in cosmological solutions and their relation with other global properties. Also, the properties of causal relations between points of space-time are carried out. It is shown that these could be used to determine physically the manifold structure of space-time if the strong causality assumption holds. The concepts of a null horizon and a partial Cauchy surface are introduced. He also studied the physical significance of curvature using the deviation equation for timelike and null curves. Further, He defined singularity of space-time in terms of geodesic incompleteness.

In 2014, R. Mirzaie [40] examined the topological description of orbit spaces and orbits of some flat Lorentzian G-manifolds. Viviana del Barco et al. investigated [39] the families of compact Lorentzian manifolds in dimension four. They showed that every lightlike geodesic on such spaces is periodic, while there are closed and non-closed spacelike and timelike geodesics.

T. Turhan et al. [35] obtained some results for differential geometry of spacelike curves in 3-dimensional De Sitter space and studied the frenet frame, the frenet equations and geodesic curvature and torsion functions to analyse and characterize the shape of the curves in 3-dimensional De Sitter space. In 2013, M. Czarnecki et al. [36] investigated the behaviour of some surfaces and foliations as their families using computation in the De Sitter space.

4 Motivation

The most natural topology on Minkowski space is the Euclidean topology. Motivation for defining non-Euclidean topologies on Minkowski space comes from the facts that (i) the Euclidean topology does not take into account the causal structure of Minkowski space and (ii) the homeomorphism group of Minkowski space with Euclidean topology is too large to be of any physical significance. These topologies are t-topology, s-topology, fine topology, time topology, space topology etc.

The sequence lemma holds in a first countable space. The non-Euclidean topologies t-topology, s-topology, f-topology on Minkowski space are first countable [2, 3, 1] and hence the sequence lemma holds. A countably infinite product of R with box topology is not first countable and the sequence lemma does not hold [25]. R^n with the cofinite topology is a non-first countable space where sequence lemma holds. This shows that the sequence lemma may or may not hold in a non-first countable space. In this light, the lemma can be dealt in the context of non-first countable non-manifold topologies on Lorentz manifolds. Further, it is known that $R^n - A$ is disconnected, where A is a closed copy of R^{n-1} in R^n. This may lead to study the connectedness of $M^n - A$ in the context of non-manifold topologies, where A is a closed copy of M^n.

Some major results of homotopy theory include Borsuk-Ulam Theorem and Brouwer Fixed-Point Theorem concerning R^n with the Euclidean topology [25]. These results can be studied in the context of non-manifold topology. Agrawal et al. discussed fundamental group of Mikowski space with t-topology [2]. Also, zeroth singular homology group of pathwise connected space is known [8]. Further, pth singular homology group of R^n, convex subset of R^n is zero for $p >0$ and pth singular homology group of open set of R^n is zero for $p \geq n$ [9, 8]. This motivates to determine the homology group of Minkowski space with non-manifold topologies.

The $n-1$ dimensional De Sitter space is a subspace of n- dimensional Minkowski space. Topological properties of n-dimensional Minkowski space with the non-manifold topologies are known. Some of the topological properties like normality, separability, compactness, locally compactness, lindelöfness, locally connectedness are not hereditary and hence these properties

can be studied for De Sitter space with the non-manifold topologies.

5 Proposed Direction of Work

Zeeman [37] defined physically significant non-Euclidean topologies on Minkowski space which comprise t-topology, fine topology, space topology, time topology etc. The homeomorphism group of these topologies is generated by the inhomogeneous Lorentz group, dialatations and translations. Topological properties namely normality, first countability, compactness, locally compactness have been studied in the context of these topologies [2, 12, 26]. The corresponding topological spaces may not have the same topological properties. For example, t-topology is first countable, although fine topology, space topology and time topology are non-first countable.

As the sequence lemma holds in a first countable space [25], it is pertinent to find whether it holds in the context of non-first countable non-manifold topologies. In a similar way, the other results that can be explored include the Borsuk-Ulam Theorem, the Brouwer Fixed-Point Theorem, disconnectedness of $R^n - A$, where A is a closed copy of R^{n-1} in R^n etc.

Further, the homology groups of R^n and some of its sets are known [8] which can also be studied in the new setting and as the topological properties are not necessarily hereditary, these can be studied for subspaces of Lorentz manifold.

In this light, the present proposal is aimed at the study of the following:

1. Sequence Lemma in the context of non-first countable non-manifold topologies namely fine topology, space topology, time topology and A-topology.

2. Connectedness of $M^n - A$, where A is a closed copy of M^n, Brouwer Fixed-Point Theorem, Borsuk-Ulam Theorem and other similar results in the context of non-manifold topologies.

3. Homology groups of Minkowski space and its subsets with non-manifold topologies.

4. Topological properties of important subspaces namely the De Sitter space of Lorentz manifolds with non-manifold topologies.

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